MTH 161 - Lecture 7

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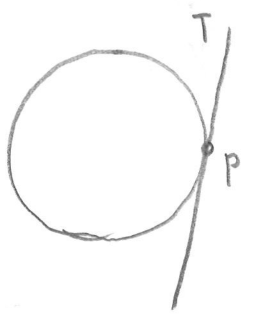
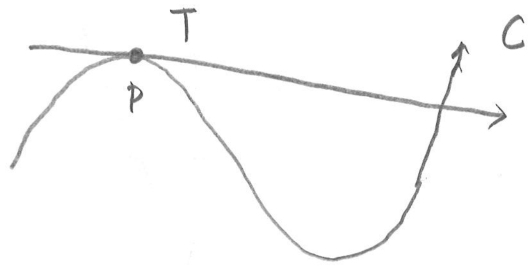
2. DERIVATIVES

Derivatives and rates of change

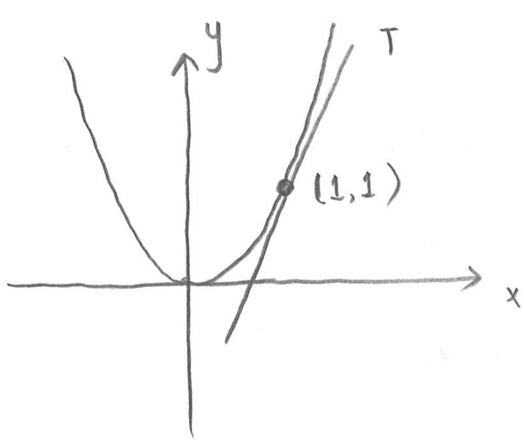
The problem of finding the tangent line to a curve and the problem of finding the velocity of an object involve finding the same type of limit, which we call derivative.

Tangent Line





Ex Find the equation of the tangent line to the parabola $y=x^2$ at the pt $(1,1)$



So we will be able find an equation of the tangent line T as soon as we Know it's slope m.

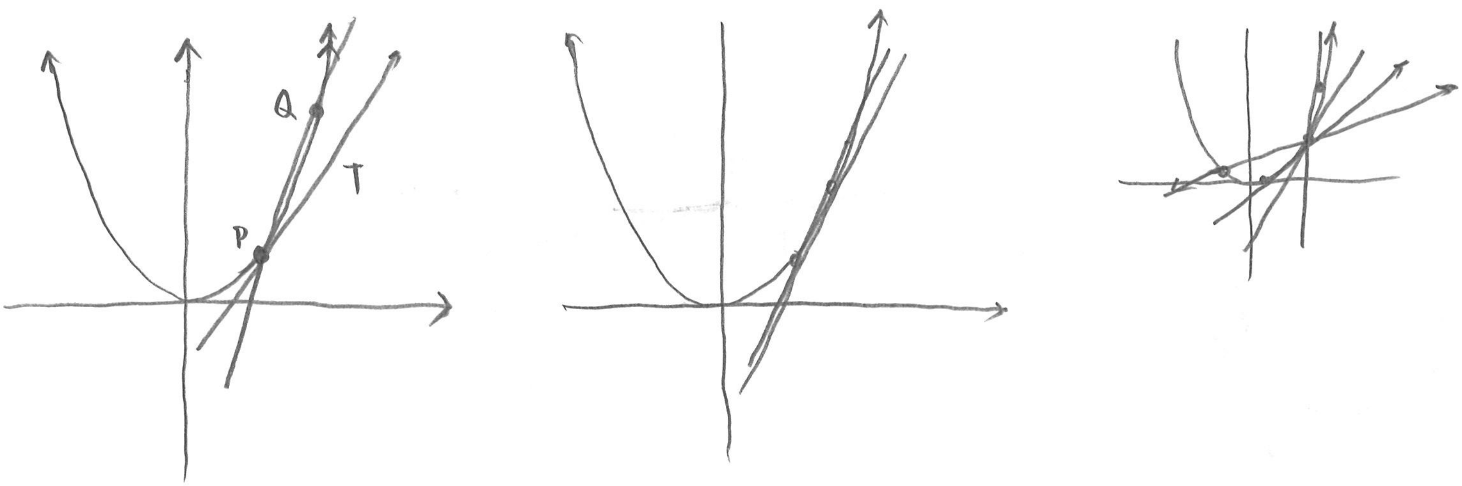
The difficulty is that we Know only one point, P on T, whereas we need two points to compute the slope.

But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola and compute the slope m_{PQ} of the secant line PQ [line that cuts (intersects) a curve more than once].

Choose $x \neq 1$ so that $Q \neq P$. Then $m_{PQ} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{x^2 - 1}{x - 1}$

What happens as x approaches 1

From both sides



So actually the slope m of the tangent line is the limit of the slopes of the secant lines as x approaches 1

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1} (x+1) = 1 + 1 = 2.$$

Then point-slope form of the eq^n of the line, $y - y_1 = m(x - x_1)$

$$y - 1 = 1(x-1) \Rightarrow y = 2x - 1.$$

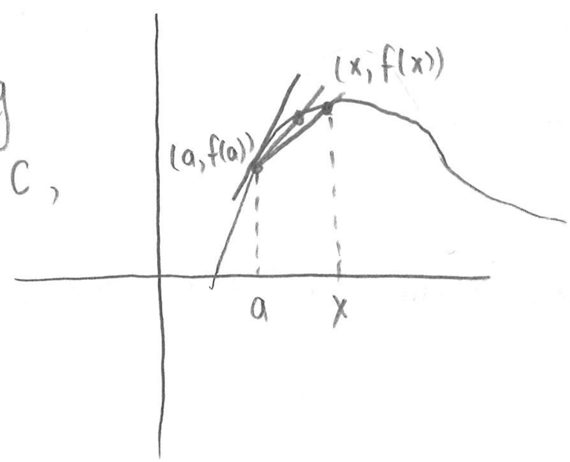
- We sometimes refer to the slope of the tangent line to a curve at a point as the slope of the curve at the point. The idea is that if we zoom in far enough toward the point, the curve almost like a straight line.

The more we zoom in, the more the curve looks like it's tangent line ☑

In general, if a curve C has eq^n $y = f(x)$, we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$ and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along C by letting x approach a. If m_{PQ} approaches m along C, then we define the tangent T to be the line through P with slope m.



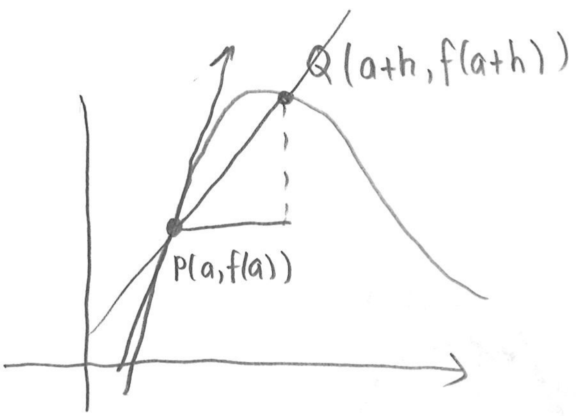
Def^n The tangent line to curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided the limit exists.

Alternate description

There is another expression for the slope of the tangent line that is sometimes easier to use.

If $h = x - a$, then $x = a + h$, and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$



Then,

$$\boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Ex Find an equation of the tangent line to hyperbola $y = \frac{3}{x}$ at the point $(3, 1)$.

Solution Let $f(x) = \frac{3}{x}$. Then the slope of the tangent at $(3, 1)$ is

$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\frac{3}{3+h} - \overset{1}{\overset{\|}{\frac{3}{3}}}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3 - (3+h)}{3+h}}{\cancel{h}} = \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(3+h)} = -\frac{1}{3}$$

Therefore eq^n of the tangent line at pt $(3,1)$ is

$$y - 1 = -\frac{1}{3}(x-3) \implies 3y - 3 = -x + 3 \implies x + 3y = 6 \qquad \square$$

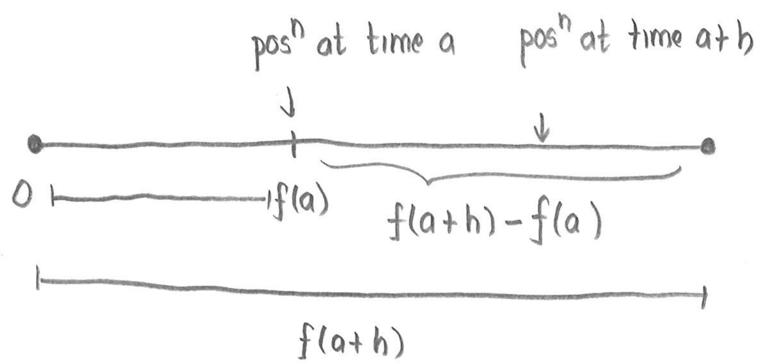
The Velocity problem

When we first studied limit, we wanted to approximate velocity of a dropped ball and we actually defined its limiting value of average velocities over shorter and shorter time periods.

In general, suppose an object moves along a straight line according to the eq^n $s = f(t)$, where s is the displacement (direct distance) of the object from the origin at time t.

The function f that describes the motion is called the position function of the object.

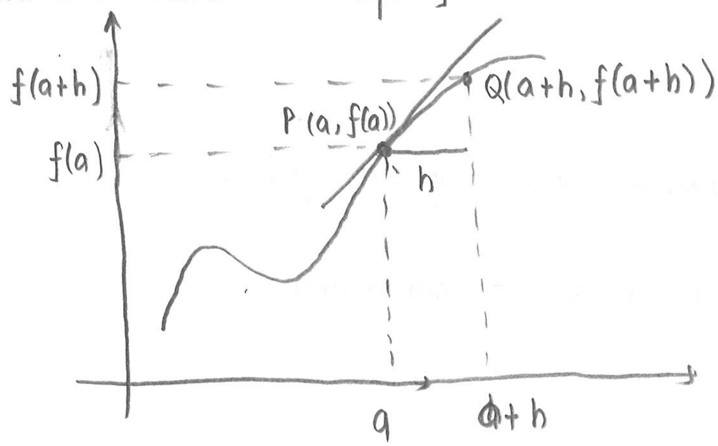
In the time interval from $t = a$ to $t = a+h$, the change in position is $f(a+h) - f(a)$



Then, the average velocity over the interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$$

which is the same as the slope of the secant line.



Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a+h]$ i.e we let h approach 0.

Then we define velocity (instantaneous velocity) $v(a)$ at time $t=a$, to be the limit of these average velocities:

$$v(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

This means that velocity at time $t=a$, is equal to the slope at the tangent line at P of the position function.

So what is the velocity of a ball dropped from the hill 1000 m above ground after 5s ?

Sol^n Equation of motion $s = f(t) = 4.9t^2$

Then velocity after 5s is,

$$v(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \to 0} \frac{4.9(5+h)^2 - 4.9(5)^2}{h}$$

$$= \lim_{h \to 0} \frac{4.9\left[\cancel{5^2} + 10h + h^2 - \cancel{5^2}\right]}{h} = \lim_{h \to 0} \frac{4.9\cancel{h}(10+h)}{\cancel{h}}$$

$$= 4.9(10) = 49 \text{ m/s}.$$

$\boxed{}$

So the formula, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ Keeps popping up everywhere whenever we calculate rate of change. Since it shows up everywhere, it is given a special name and notation.

Defn The derivative of a function f at a number a, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ if the limit exists.

Again if we write $x = a + h$, then $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore we have an equivalent definition:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Ex Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a

Soln From defn we have,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{8a} - 8h + \cancel{9} - \cancel{a^2} + \cancel{8a} - \cancel{9}}{h} = \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2a + h - 8)}{\cancel{h}} = \lim_{h \to 0} 2a + h - 8 = 2a - 8$$

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So we have that the slope of a tangent line has the exact same def^n as derivative.

Therefore, we have the following defn.

- Defn The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Then using point slope formula, we can write the eq^n of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)[x - a]$$

Ex Find the eq^n of the tangent line to the parabola $y = x^2 - 8x + 9$ at pt $(3, -6)$.

Sol^n We know, $f'(a) = 2a - 8$

So slope of tangent line at $(3, -6)$ is $f'(3) = 2 \cdot 3 - 8 = 6 - 8 = -2$

So, $y + 6 = -2(x - 3) \Rightarrow y + 6 = -2x + 6 \Rightarrow y = -2x$.

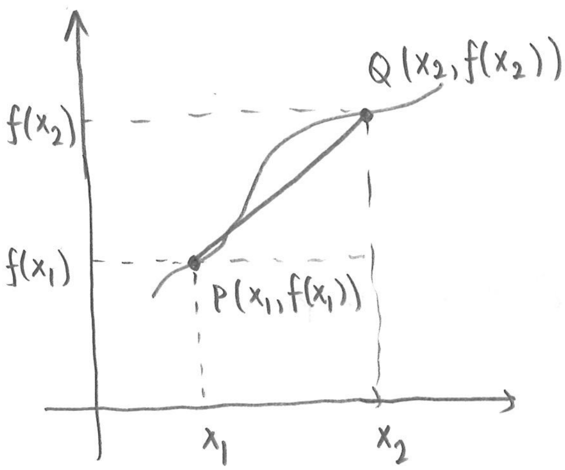
RATES OF CHANGE

Let $y = f(x)$

If x changes from x_1 to x_2, the change in x is denoted by $\Delta x = x_2 - x_1$

and the corresponding change in y is $\Delta y = f(x_2) - f(x_1)$

The difference quotient $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is called the average rate of change of y with respect to x. over the interval $[x_1, x_2]$



Can be thought of as the slope of secant line PQ.

Just like velocity, we consider average rate over shorter and shorter interval.

i.e by letting x_2 getting closer and closer to x_1, or letting Δx approach 0.

The limit of these average rate of change is called <u>instantaneous rate of change of y with respect to x at $x = x_1$</u>, which can be interpreted as the slope of the tangent to $y = f(x)$ at $x = x_1$

So,

instantaneous rate of change (of y wrt x at $x = x_1$)

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

So we have the second interpretation of derivative $f'(a)$.

Defn The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x, when $x = a$.